

Euler-Mascheroni constant and results.

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# Chapter 1

## Euler's constant in infinite sum form. (Sondow (2005).)

(Note: The index of the inner summand in the Lean definition is shifted by one so the series can be expressed as a sum from  $n = 0$  to  $\infty$  (so we can use "tsum" for cleaner proof).)

**Lemma 1** (Nonnegativity of the inner sum). *For every integer  $n \geq 1$ ,*

$$\frac{1}{n} - \ln \frac{n+1}{n} \geq 0.$$

*Proof.* The inequality  $\ln x \leq x - 1$  holds for all  $x > 0$ . Taking  $x = \frac{n+1}{n}$  gives

$$\ln \frac{n+1}{n} \leq \frac{n+1}{n} - 1 = \frac{1}{n}.$$

Rearranging yields the stated nonnegativity. □

**Lemma 2** (Partial sum of the infinite sum). *For every integer  $N \geq 1$ ,*

$$\sum_{n=1}^N \left( \frac{1}{n} - \ln \frac{n+1}{n} \right) = H_N - \ln(N+1),$$

where  $H_N = \sum_{n=1}^N \frac{1}{n}$  is the  $N$ -th harmonic number.

*Proof.* Split the sum:

$$\sum_{n=1}^N \frac{1}{n} - \sum_{n=1}^N \ln \frac{n+1}{n}.$$

The first term is  $H_N$ . The second term telescopes:

$$\sum_{n=1}^N (\ln n + 1 - \ln n) = \ln(N+1) - \ln 1 = \ln(N+1).$$

Combining gives the claimed identity. □

**Theorem 3** (Upper bound of the sum). *For every integer  $N \geq 1$ ,*

$$\sum_{n=1}^N \left( \frac{1}{n} - \ln \frac{n+1}{n} \right) \leq 1$$

*Proof.* We use the results from the previous lemma, and get the sum as  $H_N - \ln N + 1$ . This is the same as mathlib's "Real.eulerMascheroniSeq" we can use the results from mathlib. (Alternately, we can use the fact that the upper bound of  $\gamma$  is  $2/3$  and  $\gamma < H_N - \ln N + 1$  to get a stricter results?)  $\square$

**Theorem 4** (Sondow 2005 (2) (Infinite sum)).

$$\gamma = \lim_{n \rightarrow \infty} (H_n - \ln n) = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \ln \frac{n+1}{n} \right).$$

*Proof.* By combining all of the previous results (plus the fact that  $H_n - \ln n + 1$  that we got from partial sum is basically the same as mathlib's one), we can conclude that the infinite sum is equal to  $\lim_{n \rightarrow \infty} (H_n - \ln n)$   $\square$